Introduction to Mathematics and Modeling

lecture 6

Integrals
This week

The integraph, an instrument for measuring integrals

1. Section 5.1: area estimating with finite sums
2. Section 5.2: limits of finite sums
3. Section 5.3: the definite integral
4. Section 5.4: the fundamental theorem of calculus
We can write sums in a shorter way using the Σ-notation:

\[
\sum_{k=M}^{N} a_k = a_M + a_{M+1} + a_{M+2} + \cdots + a_{N-1} + a_N
\]

- Σ is the Greek letter “S” (pronounced as ’sigma’), which refers to “Sum”.

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- $a_k$ is the $k$-th term of the sum, and is a formula containing $k$.
- If $N < M$ then the sum is equal to 0 by convention.
- The index is a dummy:

$$\sum_{k=3}^{6} a_k = \sum_{p=3}^{6} a_p = a_3 + a_4 + a_5 + a_6$$
The Σ-notation

\[ \sum_{k=1}^{12} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 + 12^2 \]

\[ = 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 + 121 + 144 \]

\[ = 650. \]
Examples

\[
\sum_{k=1}^{4} (-1)^k k = (-1)^1 \cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3 + (-1)^4 \cdot 4 \\
= -1 + 2 - 3 + 4 = 2.
\]
Examples

1.3

\[ \sum_{k=1}^{4} (-1)^k k = (-1)^1 \cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3 + (-1)^4 \cdot 4 \]

\[ = -1 + 2 - 3 + 4 = 2. \]

\[ \sum_{k=1}^{2} \frac{k}{k+1} = \frac{1}{1+1} + \frac{2}{2+1} \]

\[ = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}. \]
The sum of the first $n$ positive integers is equal to $\frac{n(n + 1)}{2}$.

With $\sum$-notation:

$$1 + 2 + \cdots + (n - 1) + n = \sum_{k=1}^{n} k = \frac{n(n + 1)}{2}.$$
Theorem

The sum of the first $n$ positive integers is equal to $\frac{n(n + 1)}{2}$.

- with $\Sigma$-notation:

$$1 + 2 + \cdots + (n - 1) + n = \sum_{k=1}^{n} k = \frac{n(n + 1)}{2}.$$

- Write out the terms in the sum twice:

$$1 + 2 + \cdots + (n - 1) + n$$

$$n + (n - 1) + \cdots + 2 + 1$$
Arithmetic series

**Theorem**

The sum of the first \( n \) positive integers is equal to \( \frac{n(n + 1)}{2} \).

- with \( \Sigma \)-notation:
  \[
  1 + 2 + \cdots + (n - 1) + n = \sum_{k=1}^{n} k = \frac{n(n + 1)}{2}.
  \]

- Write out the terms in the sum *twice*:
  \[
  \begin{array}{cccccccc}
  1 & + & 2 & + & \cdots & + & (n - 1) & + & n \\
  n & + & (n - 1) & + & \cdots & + & 2 & + & 1 \\
  \hline
  (n + 1) & + & (n + 1) & + & \cdots & + & (n + 1) & + & (n + 1)
  \end{array}
  \]

- Adding the columns gives \( n \) terms, all equal to \( n + 1 \), so
  \[
  2 \sum_{k=1}^{n} k = n(n + 1).
  \]
Sum- and difference rule:

\[
\sum_{k=M}^{N} (a_k + b_k) = \sum_{k=M}^{N} a_k + \sum_{k=M}^{N} b_k, \quad \text{and} \quad \sum_{k=M}^{N} (a_k - b_k) = \sum_{k=M}^{N} a_k - \sum_{k=M}^{N} b_k.
\]
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  \]

- **Constant multiple rule:**
  \[
  \sum_{k=M}^{N} c \cdot a_k = c \sum_{k=M}^{N} a_k.
  \]
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\[
\sum_{k=M}^{N} c a_k = c \sum_{k=M}^{N} a_k.
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- **Constant value rule:**

\[
\sum_{k=M}^{N} c = (N - M + 1) c.
\]
- **Sum- and difference rule:**
  \[ \sum_{k=M}^{N} (a_k + b_k) = \sum_{k=M}^{N} a_k + \sum_{k=M}^{N} b_k, \text{ and } \sum_{k=M}^{N} (a_k - b_k) = \sum_{k=M}^{N} a_k - \sum_{k=M}^{N} b_k. \]

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- **Constant value rule:**
  \[ \sum_{k=M}^{N} c = (N - M + 1) c. \]

- **Splitting rule:**
  \[ \sum_{k=M}^{N} a_k = \sum_{k=M}^{P} a_k + \sum_{k=P+1}^{N} a_k. \]
Define $S_n$ as the sum of the first $n$ odd integers:

$$S_n = 1 + 3 + \cdots + (2n - 1)$$
Example

- Define $S_n$ as the sum of the first $n$ odd integers:

$$S_n = 1 + 3 + \cdots + (2n - 1)$$

- Notice that

$$S_n + (2 + 4 + \cdots + 2n) = 1 + 2 + 3 + \cdots + (2n - 1) + 2n$$

$$= \frac{2n(2n + 1)}{2} = n(2n + 1) = 2n^2 + n$$
Example

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Furthermore

$$2 + 4 + \cdots + 2n = \sum_{k=1}^{n} 2k = 2 \sum_{k=1}^{n} k$$

$$= 2 \cdot \frac{n(n + 1)}{2} = n(n + 1) = n^2 + n.$$
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  \]

- Therefore
  \[
  S_n = (2n^2 + n) - (n^2 + n)
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$$= 2 \cdot \frac{n(n + 1)}{2} = n(n + 1) = n^2 + n.$$

Therefore

$$S_n = (2n^2 + x) - (n^2 + x) = n^2.$$
The sum of the first $n$ odd integers is equal to $n^2$:

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Note that
\[1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \sum_{k=1}^{5} k^2,\]
but also
\[1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \sum_{m=0}^{4} (m + 1)^2,\]
and even
\[1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \sum_{h=3}^{7} (h - 2)^2.\]
Shifting the index

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\[ 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \sum_{h=3}^{7} (h - 2)^2. \]

Since the index is a dummy, you could even write

\[ \sum_{k=1}^{5} k^2 = \sum_{k=0}^{4} (k + 1)^2 = \sum_{k=3}^{7} (k - 2)^2. \]
Assignment: IMM1 - Tutorial 6.1
Definition

A partition of the interval \([a, b]\) in \(n\) subintervals is a sequence \(x_0, x_1, \ldots, x_n\) constructed as follows:

\[(i)\quad \Delta x = \frac{b - a}{n}\]
\[(ii)\quad x_k = a + k\Delta x \quad (k = 0, 1, \ldots, n)\]
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(ii) \(x_k = a + k\Delta x\) \((k = 0, 1, \ldots, n)\)

Note that

\[x_0 = a, \quad x_n = b, \quad x_k - x_{k-1} = \Delta x.\]
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- Note that

\[x_0 = a, \quad x_n = b, \quad x_k - x_{k-1} = \Delta x.\]

- The number \(\Delta x\) is called the mesh of the partition.
Partition $[a, b]$ in 4 subintervals.

**Definition**

We define the 4-th Riemann sum of $f$ over $[a, b]$ as

$$\sum_{k=1}^{4} f(x_k) \cdot \Delta x \quad \text{where} \quad \Delta x = \frac{b - a}{4}$$
Partition \([a, b]\) in 8 subintervals.

**Definition**

We define the 8-th Riemann sum of \(f\) over \([a, b]\) as

\[
\sum_{k=1}^{8} f(x_k) \cdot \Delta x \quad \text{where} \quad \Delta x = \frac{b - a}{8}
\]
Partition \([a, b]\) in \(n\) subintervals.

**Definition**

We define the \(n\)-th Riemann sum of \(f\) over \([a, b]\) as

\[
\sum_{k=1}^{n} f(x_k) \cdot \Delta x \quad \text{where} \quad \Delta x = \frac{b - a}{n}
\]
Consider a moving object and assume that we know its velocity as a function of time $v(t)$.

Can we compute the displacement using the function $v(t)$?
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Can we compute the displacement using the function $v(t)$?

- If $v(t) = v_0$ is constant, then the displacement is equal to the product of $v_0$ and the elapsed time.
Computing displacement from velocity

Consider a moving object and assume that we know its velocity as a function of time \( v(t) \).

Can we compute the displacement using the function \( v(t) \)?

- If \( v(t) = v_0 \) is constant, then the displacement is equal to the product of \( v_0 \) and the elapsed time.
- If \( v(t) \) is not constant, then we approximate the displacement with a Riemann sum.
Assume that we know the velocity $v(t)$ for $a \leq t \leq b$, then we can find an estimate for the displacement while $t$ elapses from $a$ to $b$.

(1) Partition the interval $[a, b]$ in $n$ subintervals with mesh $\Delta t = \frac{b - a}{n}$ and intermediate points $t_k = a + k\Delta t$.

(2) - While $t$ runs from $a = t_0$ to $t_1$, the displacement is approximately $v(t_1)\Delta t$;
- while $t$ runs from $t_1$ to $t_2$, the displacement is approximately $v(t_2)\Delta t$;
  etcetera.

(3) The total displacement is approximately

$$\sum_{k=1}^{n} v(t_k)\Delta t.$$
Assignment: IMM1 - Tutorial 6.2

**Note:** The Riemann sums that are used in the slides are upper sums. For lower sums, choose the function value at the left boundary of all subintervals:

\[ \sum_{k=1}^{n} f(x_k) \Delta x \quad \text{where} \quad \Delta x = \frac{b - a}{n} \quad \text{and} \quad x_k = a + (k - 1)\Delta x. \]
Approximate the area of the triangle with vertices (0, 0), (1, 0) and (1, 1) with a Riemann sum.
Approximate the area of the triangle with vertices \((0, 0), (1, 0)\) and \((1, 1)\) with a Riemann sum.

Define the partition \(x_k = k\Delta x = \frac{k}{n}\) with \(\Delta x = \frac{1}{n}\).
Approximate the area of the triangle with vertices (0, 0), (1, 0) and (1, 1) with a Riemann sum.

- Define the partition \( x_k = k\Delta x = \frac{k}{n} \) with \( \Delta x = \frac{1}{n} \).
- The Riemann sum of \( f(x) = x \) is

\[
\sum_{k=1}^{n} x_k \Delta x = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n}.
\]
Evaluate the Riemann sum:

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\[
\sum_{k=1}^{n} x_k \Delta x = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \sum_{k=1}^{n} \frac{k}{n^2}
\]
Evaluate the Riemann sum:

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\sum_{k=1}^{n} x_k \Delta x = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2} \sum_{k=1}^{n} k
\]
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\]

\[
= \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{1}{2} \left(1 + \frac{1}{n}\right) = \frac{1}{2} + \frac{1}{2n}.
\]
Evaluate the Riemann sum:

\[ \sum_{k=1}^{n} x_k \Delta x = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{1}{2} \left( 1 + \frac{1}{n} \right) = \frac{1}{2} + \frac{1}{2n}. \]

If we let \( n \) approach infinity then

\[ \lim_{n \to \infty} \sum_{k=1}^{n} x_k \Delta x = \frac{1}{2}. \]
■ Evaluate the Riemann sum:
\[
\sum_{k=1}^{n} x_k \Delta x = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2} \sum_{k=1}^{n} k \\
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\[ = \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{1}{2} \left( 1 + \frac{1}{n} \right) = \frac{1}{2} + \frac{1}{2n}. \]

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\]

If we let \( n \) approach infinity then

\[
\lim_{n \to \infty} \sum_{k=1}^{n} x_k \Delta x = \frac{1}{2}.
\]
For a positive function, a Riemann sum can be regarded as the approximation of the surface area of the region $R$ bounded by the graph of $f$, the $x$ axis, and the lines $x = a$ and $x = b$.

**Definition**

The **definite integral of $f$ over the interval** $[a, b]$ is defined as

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \left( \sum_{k=1}^n f(x_k) \cdot \Delta x \right)$$

A definite integral can be regarded as the area of the region $R$. 
The variable in the integral is a *dummy*:

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(u) \, du
\]
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\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(u) \, du
\]

Linearity:

\[
\int_{a}^{b} \alpha f(x) + \beta g(x) \, dx = \alpha \int_{a}^{b} f(x) \, dx + \beta \int_{a}^{b} g(x) \, dx
\]
Laws of integration

- The variable in the integral is a *dummy*:

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(u) \, du
\]

- Linearity:

\[
\int_{a}^{b} \alpha f(x) + \beta g(x) \, dx = \alpha \int_{a}^{b} f(x) \, dx + \beta \int_{a}^{b} g(x) \, dx
\]

- Additivity:

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx
\]
The variable in the integral is a \textit{dummy}:

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\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(u) \, du
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Linearity:

\[
\int_{a}^{b} \alpha f(x) + \beta g(x) \, dx = \alpha \int_{a}^{b} f(x) \, dx + \beta \int_{a}^{b} g(x) \, dx
\]

Additivity:

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx
\]

Interchanging the upper and lower limit gives a minus sign:

\[
\int_{a}^{b} f(x) \, dx = - \int_{b}^{a} f(x) \, dx
\]
If a function is negative on an interval, then the integral over that interval is negative.
If a function is negative on an interval, then the integral over that interval is negative.

The integral adds the areas of the positive part of $f$, but subtracts the areas of the negative parts:

$$\int_a^b f(x) \, dx = \text{Area}(A) - \text{Area}(B).$$
Constant functions

\[ \int_a^b c \, dx = c(b - a) \]

Notice that the Riemann sum of any partition is

\[ \sum_{k=1}^{n} c \Delta x = n \cdot c \Delta x = c \frac{b - a}{n} = c(b - a). \]
\[
\int_a^b x \, dx = \frac{1}{2} b^2 - \frac{1}{2} a^2
\]
Laws of integration

\[ \int_{a}^{b} x \, dx = \frac{1}{2} b^2 - \frac{1}{2} a^2 \]

- Note that \( \frac{1}{2} b^2 - \frac{1}{2} a^2 = \frac{1}{2} (b + a)(b - a) = \text{Area}(R). \)
Assignment: IMM1 - Tutorial 6.3
Differentiate displacement to compute velocity:

\[ v(t) = s'(t) \]
Differentiate displacement to compute velocity:

\[ v(t) = s'(t) \]

The displacement can be computed from the velocity by integrating:

\[ s(t) = \lim_{n \to \infty} \sum_{k=1}^{n} v(t_k) \Delta t = \int_{0}^{t} v(\tau) \, d\tau \]

The integral \( \int_{0}^{t} v(\tau) \, d\tau \) is a function \( s(t) \) whose derivative is \( v \).
Add the charges in all compartments:

\[ Q(t) = \sum_{k=1}^{n} i(t_k) \Delta t. \]
Add the charges in all compartments:

\[ Q(t) = \sum_{k=1}^{n} i(t_k) \Delta t. \]

The total charge passing through \( A \) is

\[ Q(t) = \lim_{n \to \infty} \sum_{k=1}^{n} i(t_k) \Delta t = \int_{0}^{t} i(\tau) \, d\tau. \]
Add the charges in all compartments:

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Current is the rate of change of charge:

\[ i(t) = Q'(t). \]
**Definition**

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Antiderivatives

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Theorem

Let $(x_0, y_0)$ be a point in the plane. Then there is a unique antiderivative $F$ of $f$ for which $F(x_0) = y_0$.
Let \( f(x) = e^x + 1 \), then \( F(x) = e^x + x \) is an antiderivative of \( f \).
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For arbitrary $C$ the function

$$F_C(x) = e^x + x + C$$

is also an antiderivative of $f$. 

\[ F(0) = 4 = e^0 + 0 + C \]

Hence $C = 3$. 

\[ F(x) = e^x + x + 3 \]
Example

Let $f(x) = e^x + 1$, then $F(x) = e^x + x$ is an antiderivative of $f$.

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There is only one antiderivative of $f$ for which $F(0) = 4$:

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There is only one antiderivative of \( f \) for which \( F(0) = 4 \):
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F(x) = e^x + x + 3.
\]
The correct value for \( C \) is found by solving the equation \( F_c(0) = 4 \):
\[
4 = F_c(0) = e^0 + 0 + C = 1 + C,
\]
hence \( C = 3 \).
The Fundamental Theorem of Calculus

1. Define the function
   \[ F(x) = \int_a^x f(t) \, dt, \]
   then \( F \) is an antiderivative for \( f \), in other words: \( F'(x) = f(x) \).

2. If \( F \) is an antiderivative for \( f \) then
   \[ \int_a^b f(t) \, dt = F(b) - F(a). \]
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- The function \( F(x) = \int_a^x f(t) \, dt \) also satisfies \( F(a) = 0 \), so \( F \) is the unique antiderivative of \( f \) for which \( F(a) = 0 \).
The inverse of differentiation

$$F(x) = \int_a^x f(t) \, dt$$

$$F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, dt = f(x)$$
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Integrals with $\sin$ and $\cos$

\[ \int_a^b f(x) \, dx = F(x) \bigg|_a^b = F(b) - F(a) \quad \text{where } F' = f. \]

\[
\int_0^{\pi/2} \sin(x) \, dx = -\cos(x) \bigg|_0^{\pi/2} \\
= -\cos\left(\frac{\pi}{2}\right) - (-\cos 0) \\
= -0 - (-1) = 1
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Integrals with \(\sin\) and \(\cos\)

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\int_a^b f(x) \, dx = F(x) \bigg|_a^b = F(b) - F(a) \quad \text{where } F' = f.
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\[\int_\pi^{2\pi} \sin(x) \, dx = -\cos(x) \bigg|_\pi^{2\pi} = -\cos(2\pi) - (-\cos(\pi)) = -1 - (-(-1)) = -2\]
The antiderivative of $\cos(x)$

$$\int_0^x \cos(t) \, dt = \sin(t) \bigg|_{t=0}^x = \sin(x).$$
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The antiderivative of $\sin(x)$

$$\int_0^x \sin(t) \, dt = -\cos(t) \bigg|_{t=0}^{x} = -\cos(x) - (-1) = -\cos(x) + 1.$$
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Notice that for arbitrary real $\alpha$ we have

$$\frac{d}{dx} \left( x^{\alpha+1} \right) = (\alpha + 1)x^\alpha.$$
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Hence, if $\alpha \neq -1$:

$$\frac{d}{dx} \left( \frac{1}{\alpha+1} x^{\alpha+1} \right) = x^\alpha.$$
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The antiderivative of $x^\alpha$ is:

$$\frac{1}{\alpha + 1} x^{\alpha+1} + C \quad \text{if} \quad \alpha \neq -1.$$
Notice that for arbitrary real $\alpha$ we have
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\frac{1}{\alpha + 1} x^{\alpha+1} + C \quad \text{if } \alpha \neq -1.
\]

The antiderivative of $x^{-1} = \frac{1}{x}$ is:
\[
\ln |x| + C.
\]

See lecture 5
\[
\int_{0}^{1} (2x^3 - 2x + 1) \, dx
\]
\[
\int_0^1 (2x^3 - 2x + 1) \, dx = \int_0^1 2x^3 \, dx - \int_0^1 2x \, dx + \int_0^1 1 \, dx
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\[ \int_0^1 2x^3 - 2x + 1 \, dx = \int_0^1 2x^3 \, dx - \int_0^1 2x \, dx + \int_0^1 1 \, dx \]

\[ = \left[ \frac{1}{2} x^4 \right]_0^1 - \left[ x^2 \right]_0^1 + \left[ x \right]_0^1 \]
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\int_0^1 2x^3 - 2x + 1 \, dx = \int_0^1 2x^3 \, dx - \int_0^1 2x \, dx + \int_0^1 1 \, dx
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\]

\[
= \left( \frac{1}{2} \cdot 1^4 - \frac{1}{2} \cdot 0^4 \right) - (1^2 - 0^2) + (1 - 0)
\]
\[
\int_0^1 2x^3 - 2x + 1 \, dx = \int_0^1 2x^3 \, dx - \int_0^1 2x \, dx + \int_0^1 1 \, dx
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= \left[ \frac{1}{2} x^4 \right]_0^1 - \left[ x^2 \right]_0^1 + \left[ x \right]_0^1
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= \left( \frac{1}{2} \cdot 1^4 - \frac{1}{2} \cdot 0^4 \right) - (1^2 - 0^2) + (1 - 0)
\]

\[
= \frac{1}{2} - 1 + 1 = \frac{1}{2}.
\]
Assignment: IMM1 - Tutorial 6.4